sharpened Hamiltonian Numbers E_{n+1}, E_n, E_{n-1}, ... and that consequently the relation—

$$E_{n+1} = 1 + \frac{E_n(E_n-1)}{1 \cdot 2} - \frac{E_{n+1}(E_{n-1}-1)(E_{n-1}-2)}{1 \cdot 2 \cdot 3} + \dots.$$

may be written in the form—

$$p = \tfrac{1}{2} + \frac{q(2q-1)}{2} - \frac{r(2r-1)(2r-2)}{2 \cdot 3} + \frac{s(2s-1)(2s-2)(2s-3)}{2 \cdot 3 \cdot 4}$$
$$- \frac{t(2t-1)(2t-2)(2t-3)(2t-4)}{2 \cdot 3 \cdot 4 \cdot 5}$$
$$+ \frac{u(2u-1)(2u-2)(2u-3)(2u-4)(2u-5)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$
$$- \dots \qquad (2.)$$

The comparison of this value of p with that given by (1) furnishes us with an equation which, after several reductions have been made in which special attention must be paid to the order of the quantities under consideration, ultimately leads to the determination of the values of A, B, C, ... in succession.

III. "Hydraulic Problems on the Cross-sections of Pipes and Channels." By Henry Hennessy, F.R.S., Professor of Applied Mathematics and Mechanism in the Royal College of Science for Ireland. Received March 14, 1888.

In that division of hydromechanics which is devoted to the investigation of the flow of liquids through pipes and open channels, the resistance due to the friction of the contained liquid against the sides of the pipes or channels has led to expressions for the velocity as a function of the dimensions and shape of the cross-section commonly designated as the hydraulic mean depth.

This quantity is defined as the quotient of the area of the cross-section of the liquid by that part of its perimeter in contact with the pipe or channel. In a full pipe this perimeter is identical with that of the pipe's cross-section, and in practice this is generally a circle.

It is also proved from the Calculus of Variations that a circle is the closed curve which, under a given perimeter, has the largest area, and by the same processes of analysis a segment of a circle appears to be that which includes the greatest area between its arc and its chord.

If we call the hydraulic mean depth of a pipe or channel bounded by a curved outline u, its definition gives the condition

$$u = \frac{\int y\,dx}{\int dx \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}},$$

where the limits of the integrals are taken between the same points on the curve.

If $l = \int dx \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$ is given, then the problem is to find the curve which makes $\int y\,dx$ a maximum for the given value of l. This is a well-known isoperimetrical problem*, for by the principles of the Calculus of Variations we have in this case—

$$\delta \int \left(y + \alpha \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx = 0,$$

where α is arbitrary, and therefore

$$\frac{1}{\alpha} - \frac{d}{dx}\left(\frac{dy/dx}{\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}}\right) = 0,$$

which gives

$$\frac{x - c}{\alpha} = \frac{dy/dx}{\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}}, \quad \alpha\frac{dy}{dx} = \frac{x - c}{\sqrt{\left\{1 - \left(\frac{x - c}{\alpha}\right)^2\right\}}},$$

and $y - c' = \sqrt{\{\alpha^2 - (x - c)^2\}}$ the equation of a circle with radius $= \alpha$. This result proves that for a full pipe the circle gives the greatest hydraulic mean depth, but it does not tell what is the particular arc of a circle which gives the greatest quotient for the area of the segment between itself and its chord divided by itself. This is best done by the ordinary methods of maxima and minima as follows :—

Let θ represent the angle subtended at centre by the segment of the circle whose radius in r, then—

$$u = \tfrac{1}{2}r\left(1 - \frac{\sin\theta}{\theta}\right)$$

* In his 'History of the Calculus of Variations,' p. 69, Todhunter has made a remark on this problem; namely, that if the curve instead of being closed were required to pass through two given fixed points with the arc between these points of a given length, the constants of integration would not be arbitrary, and there would be two equations from the fact of the circle passing through the given points and another arising from the given length. The solution here given avoids the necessity of two such equations by employing the well-known properties of an arc of a circle and its included segment.—March 29, 1888.

$$\frac{du}{d\theta} = \tfrac{1}{2}r\left(\frac{\sin\theta - \theta\cos\theta}{\theta^2}\right)$$

$$\frac{d^2u}{d\theta^2} = \frac{r}{2\theta^3}\left[(\theta^2 - 2)\sin\theta + 2\theta\cos\theta\right].$$

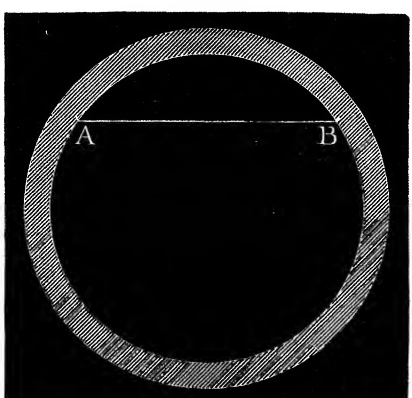
When $\dfrac{du}{d\theta} = 0, \qquad \theta = \tan\theta,$

this may be satisfied either by $\theta = 0$, or by some arc between π and 2π. The root $\theta = 0$, substituted in the value of $d^2u/d\theta^2$, makes this positive and equal to $\frac{1}{2}r$, as may be easily shown by expanding $\sin\theta$ and $\cos\theta$. Let now $\theta = \pi + \beta$, and by successive trials we shall find that $\beta = 77° \; 27'$ nearly satisfies the equation $\pi + \beta = \tan\beta$. With this value θ is $257° \; 27'$, $\cos\theta$ and $\sin\theta$ are both negative and $d^2u/d\theta^2$ is also negative, showing that the result gives a maximum for u, which in this case becomes

$$u = \tfrac{1}{2}\,r\,(1 + 0{\cdot}21722) = 0{\cdot}6086\,r, \text{ nearly.}$$

The hydraulic mean depth of a full pipe or of a half-full pipe of circular section is $0{\cdot}5\,r$, hence that for a section less by about three-tenths of the perimeter of the circle is greater. The area of the section of greatest hydraulic mean depth is $2{\cdot}74142\,r^2$ or $0{\cdot}87169$ of the entire circle. If the pipe is nearly horizontal the quantity of liquid contained in it is proportional to the cross-section, hence a circular pipe under such condition has the greatest hydraulic mean depth when it is nearly seven-eighths full; or when the liquid has fallen from the full state so as to have its free surface AB the chord of an arc of $102° \; 33'$. The versed sine of this arc is $0{\cdot}1872$ D nearly, D being the

Fig. 1.



diameter, so that for a pipe of 2 feet internal diameter the greatest hydraulic mean depth would be when the surface of the liquid had fallen below the top by 4·4928, or nearly $4\frac{1}{2}$ inches. As the velocity of the liquid is nearly as the square root of the hydraulic mean depth, the pipe filled to this height would carry liquid with a velocity slightly greater than when completely full. This conclusion is only true when the effective head of liquid is due solely to the inclination of the pipe. When the level of the liquid within the pipe falls the hydraulic mean depth tends towards its minimum value, and its decrease becomes rapid as the arc diminishes; thus if θ is a very small angle

$$u = \tfrac{1}{2} r \left(1 - \frac{\sin\theta}{\theta}\right) = \frac{r\theta^2}{12}\left(1 - \frac{\theta^2}{20}\right).$$

But $r = \mathrm{L}/\pi$, where L is the length of an arc of a semicircle; hence if the 4th power of θ is negligible we have $u = \mathrm{L}\theta^2/12\pi$.

Although pipes and conduits for water supply are usually quite full, those for drainage purposes are most commonly only partly filled with liquid, and the amount of liquid is liable to great fluctuations. This has led to the adoption for drainage pipes of an oval curve for the outline of cross-section, with the longer axis of the oval vertical and terminated at bottom by an arc of greater curvature than at top. The form of this cross-section suggests an inquiry as to how far a curve which has been often treated in isoperimetrical problems would satisfy the conditions for giving a favourable hydraulic mean depth in an open channel with fluctuating contents. We have seen that a particular arc of a circle gives a maximum for the quotient of the area of the segment divided by the perimeter of the arc, and we shall find that there is a particular catenary which gives a maximum for a corresponding quotient of the area included between its perimeter and its chord.

If as usual we make the directrix the axis of x, a the parameter, and l the length of the curve, then adopting the usual notation

$$x = a \log\left(\frac{y + \sqrt{(y^2 - a^2)}}{a}\right), \qquad \text{and } y = \sqrt{(l^2 + a^2)},$$

but in this case, as the area whose quotient divided by the perimeter is to be a maximum is the difference between the rectangle under the coordinates x and y and the area included between the curve, its parameter, and the directrix, we have manifestly—

$$u = \frac{xy - \int y\,dx}{l},$$

and as $\int y\,dx = al$, this may be written

$$u = \frac{a\sqrt{(l^2 + a^2)}\log\left(\dfrac{l + \sqrt{(l^2 + a^2)}}{a}\right) - al}{l}.$$

The shape of the curve depends on the relation between its parameter and its length, hence we must find the value of a which makes u a maximum in the above expression. The problem seems therefore to amount to this elementary statical question:—A flexible and uniform chain is attached to two supports on the same horizontal line; required the distance between the supports so as to make the area of the surface included between the chain and the horizontal line the greatest possible; or given the perimeter of a catenary to find the chord, so that the area between itself and the curve shall be a maximum. The above expression gives—

$$l\frac{du}{da} = \sqrt{(l^2 + a^2)}\log\left(\frac{l + \sqrt{(l^2 + a^2)}}{a}\right)$$
$$+ \frac{a^2}{\sqrt{(l^2 + a^2)}}\log\left(\frac{l + \sqrt{(l^2 + a^2)}}{a}\right)$$
$$+ a\sqrt{(l^2 + a^2)}\frac{d}{da}\left[\log\left(\frac{l + \sqrt{(l^2 + a^2)}}{a}\right)\right] - l$$
$$= \frac{l^2 + 2a^2}{\sqrt{(l^2 + a^2)}}\log\left(\frac{l + \sqrt{(l^2 + a^2)}}{a}\right) - 2l.$$

$$l\frac{d^2u}{da^2} = \frac{(3l^2 + 2a^2)\,a^2\log\left(\dfrac{l + \sqrt{(l^2 + a^2)}}{a}\right) - l\,(l^2 + 2a^2)\sqrt{(l^2 + a^2)}}{a\,(l^2 + a^2)^{3/2}}$$

If we write $z = l/a$, and make $du/da = 0$, we have

$$\log\left(z + \sqrt{(1 + z^2)}\right) = \frac{2z\sqrt{(1 + z^2)}}{2 + z^2}.$$

By successive trials this equation may be satisfied by substituting $z = 2{\cdot}4$, whence $l = 2{\cdot}4a$. This value substituted in the expression for d^2u/da^2 gives a negative result, and therefore the value of u is a maximum when $a = \frac{5}{12}l$. When $z = 2{\cdot}4$, $\sqrt{(1 + z^2)} = 2{\cdot}6$, and $\log\left(z + \sqrt{(1 + z^2)}\right) = \log 5 = 1{\cdot}6094$ nearly.

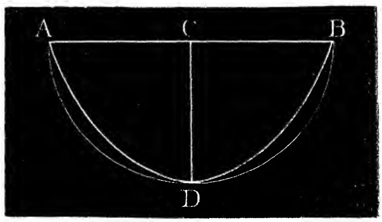
$$\frac{2z\sqrt{(1 + z^2)}}{2 + z^2} = \frac{4{\cdot}8\,.\,2{\cdot}6}{7{\cdot}76} = 1{\cdot}6082.$$

With further approximation we should find therefore—

$$x = \frac{l}{2{\cdot}4}\log 5 = \frac{2}{3}l \text{ nearly}, \qquad y = \frac{2{\cdot}6}{2{\cdot}4}l = \frac{13}{12}l.$$

But the depth h of the curve below its chord is $y - a$ or $h = \frac{2}{3}l$. In this inquiry l is the perimeter of the half curve, so that the total perimeter, the chord, and the depth are respectively in the ratio of the numbers 3, 2, and 1, or the chord of the catenary of greatest area for a given perimeter is twice the depth, and the length of the curve is three times the depth. The outline of this curve is readily shown by attaching a fine chain of 3 units of length to supports at a distance

FIG. 2.



The chord AB = 2CD.
The arc ADB of the catenary = 3CD.

of 2 units. The catenary which would give a maximum hydraulic mean depth for an open channel is therefore one whose depth is the radius and chord the diameter of a semicircle. On substituting the value of a found above in the equation for u, we shall find that the hydraulic mean depth of the catenary under consideration is nearly $0{\cdot}31l$ or $0{\cdot}155$L, where L is the total perimeter of the curve. In a semicircle the hydraulic mean depth is $\frac{1}{2}r = \mathrm{L}/2\pi$, or $0{\cdot}159$L nearly, hence the hydraulic mean depth of the catenary of maximum area is nearly equal to that of a semicircle of equal perimeter. But a channel formed by the outline of such a catenary would when the contained liquid falls, not be liable to so rapid a reduction of hydraulic mean depth as in the semicircle. For small arcs of a circle it has been shown that this is proportional to the square of the angle subtended at the centre. In the catenary if θ is the angle made by the tangent with the directrix, it is also the angle made by the radius of curvature with the axis of y, which in this case coincides with the axis of depth, and as

$$y = a \sec\theta, \quad x = a \log(\sec\theta + \tan\theta), \quad l = a\tan\theta,$$

$$u = \frac{a}{\tan\theta}\left[\sec\theta \log(\sec\theta + \tan\theta) - \tan\theta\right]$$

$$= \frac{a}{\sin\theta}\left[\log\tan\tfrac{1}{2}(\pi + \theta) - \sin\theta\right]$$

$$= a\left[\operatorname{cosec}\theta \,.\, \log\tan\tfrac{1}{2}(\pi + \theta) - 1\right]$$

$$= a\left[2\operatorname{cosec}\theta\left(\tan\tfrac{1}{2}\theta + \tfrac{1}{3}\tan^3\tfrac{1}{2}\theta + \&c. - 1\right)\right.$$

$$= a\left[\left(\frac{2}{\theta} + \frac{\theta}{3} + \quad\right)\left(\tfrac{1}{2}\theta + \frac{\theta^3}{24}\right) - 1\right]$$

$$= \frac{a\theta^2}{4},$$

when θ^3 and higher powers are omitted; and remembering that $a = l/(2{\cdot}4) = \mathrm{L}/(4{\cdot}8)$, we may write for such an arc—

$$u = \mathrm{L}\theta^2/(19{\cdot}2).$$

An arc of the semicircle at its base subtending the angle θ has when θ is small the value $\mathrm{L}\theta^2/12\pi$, as already pointed out. Hence for a circular channel and for one formed by a catenary of equal perimeter and maximum area, the hydraulic mean depth for small segments subtending equal angles would be greater for the latter. On looking at the outline of such a catenary inscribed in a semicircle, this result seems to be confirmed, and the curve approaches the oval which experience has led engineers to adopt for the section of pipes carrying fluctuating quantities of liquid.

The general result of the preceding inquiry may be summed up in the following conclusions:—For all pipes and conduits employed to convey liquid for consumption or for milling power, the circular section is the best, as the level of the liquid in the pipe is rarely, if ever, below half the diameter.

For drainage such a form is also the best if the liquid rarely falls below half the diameter, but if it is liable to fall nearly to the bottom of the pipe or conduit, an oval form, such as that actually recommended, is the best. If the pipe is likely to be as often half full as slightly filled, it is probable that some advantage would be gained by employing the catenary of maximum area for a given perimeter for the lower part of the oval. A pattern for this form can be always readily constructed by remembering the relations 1, 2, 3 for the depth, the chord, and the length of the curve. In designing the base of the pipe, it is only necessary, as already pointed out, to hang a fine chain of 3 units between supports placed at 2 units on the same horizontal line.

It is well known that in a triangular notch or triangular channel, the sides of which are at right angles, the velocity of the liquid varies but little with the depth, and it is possible to conceive that a channel may have such a form as to make such a variation extremely small.

If we suppose the surface of the liquid in an open channel to be bounded by the chord of the cross-section of the channel, then we shall have as before the hydraulic mean depth—

$$u = \frac{xy - \int y dx}{\int dx \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}},$$

and if we make u = constant—

$$x \frac{dy}{dx} \int dx \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = (xy - \int y dx) \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

the limits of the integrals in both cases being taken on the same points of the curve.

From this it follows that—

$$x \frac{dy}{dx} = c \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

which on integrating gives

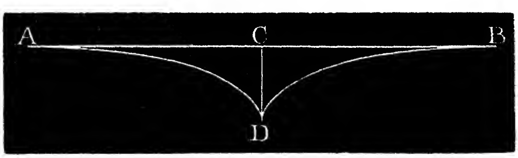
$$y = c \log \{x + \sqrt{(x^2 - c^2)}\} + C.$$

This result indicates a catenary with its convexity turned to the chord and to the axis of y, but between the limits $x = 0$ and $x = x$ the value of y becomes imaginary, the constant c being the hydraulic mean depth, which must be very small in such a case as here supposed, if we take x from $x = c$ to $x = x$

$$y = c \log \left(\frac{x + \sqrt{(x^2 - c^2)}}{c}\right),$$

and such a notch or channel might be approximately realised by two arcs of a catenary with parameters corresponding to the small arbitrary value of c.

Fig. 3.



A notch or channel with such a cross-section would have an almost constant hydraulic mean depth, but it would be inapplicable to any useful purposes in the application of hydraulics.

The cross-sections of rivers and navigable canals are regarded chiefly with reference to permanence, and the question of their hydraulic mean depth is less important than in the case of water

supply and drainage pipes. In canals the trapezoidal section is that which experience has almost universally established as the best wherever canals are carried through ordinary earth, and the rectangular section is only adopted when the sides are composed of coherent matter such as rock or masonry. The semicircular section for an open channel would not approximate to the shapes usually adopted in canals, but it may be worth remarking that the outline of the catenary of greatest area approaches more nearly to such shapes.

IV. "On the Heating Effects of Electric Currents. No. III." By W. H. PREECE, F.R.S. Received March 15, 1888.

I have taken a great deal of pains to verify the dimensions of the currents, as detailed in my paper read on December 22, 1887, required to fuse different wires of such thicknesses that the law

$$C = ad^{3/2}$$

is strictly followed; and I submit the following as the final values of the constant "a" for the different metals:—

	Inches.	Centimetres.	Millimetres.
Copper	10,244	2,530	80·0
Aluminium	7,585	1,873	59·2
Platinum	5,172	1,277	40·4
German silver	5,230	1,292	40·8
Platinoid	4,750	1,173	37·1
Iron	3,148	777·4	24·6
Tin	1,642	405·5	12·8
Alloy (lead and tin 2 to 1)	1,318	325·5	10·3
Lead	1,379	340·6	10·8

With these constants I have calculated the two following tables, which I hope will be found of some use and value:—

Fig. 1.

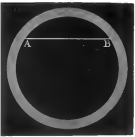


Fig. 2.

The chord AB = 2CD.
The arc ADB of the catenary = 3CD.

Fig. 3.